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NONLINEAR DIFFERENTIAL EQUATIONS AND LEGENDRE POLYNOMIALS

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Abstract. In this paper, we study non-linear differential equations associated with Legendre polynomials and their applications. From our study of non-linear differential equations, we derive some new and explicit identities for Legendre polynomials.

1. Introduction

The Legendre differential equation is given by

$$(1.1)\qquad \left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0,\quad (\text{see } [2,\ 4,\ 34]).$$

The equation (1.1) is equivalent to

$$(1.2)\qquad \frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+n\left(n+1\right)y=0,\quad (\text{see } [7,\ 21]).$$

The Legendre polynomials (or Legendre functions) are defined as the solutions of Legendre differential equation.

The generating function of Legendre polynomials $p_n(x)$ is given by

$$(1.3)\qquad \frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}p_n\left(x\right)t^n,\quad (\text{see } [1,\ 2,\ 34]).$$

In physics, the generating function of Legendre polynomials is the basis for multiple expansion.

It is known that the Rodrigues' formula of Legendre polynomials is given by

$$(1.4)\qquad p_n\left(x\right)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right],\quad (\text{see } [34]).$$

Thus, from (1.4), we note that $p_n(x)$ are polynomials of degree n.

The Legendre polynomials $p_n(x)$ can also be represented by the contour integral as

$$(1.5)\qquad p_n\left(x\right)=\frac{1}{2\pi i}\oint\left(1-2tx+t^2\right)^{-\frac{1}{2}}t^{-n-1}dt,\quad (\text{see } [2,\ 21]),$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

The first few Legendre polynomials are

$$p_0\left(x\right)=1,\quad p_1\left(x\right)=x,\quad p_2\left(x\right)=\frac{1}{2}\left(3x^2-1\right),$$

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$$p_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$$
$$p_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \quad p_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right),$$
$$p_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right), \quad \cdots.$$

As is well known, the double factorial of a positive integer n is a generalization of the usual factorial $n!$ defined by Arfken and given by

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n > 0 \text{ odd}, \\ n(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n > 0 \text{ even}, \\ 1 & \text{if } n = -1, 0, \quad (\text{see [21]}) \end{cases}. \tag{1.6}$$

As was shown by Arfken, the double factorial can be expressed in terms of the gamma function

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad (\text{see [8, 19, 21]}). \tag{1.7}$$

Thus, we note that the double factorial can also be extended to negative odd integers using the definition:

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = \frac{(-1)^n 2^n n!}{(2n)!}. \tag{1.8}$$

Now, we define the higher-order Legendre polynomials as follows:

$$\left(\frac{1}{\sqrt{1-2tx+t^2}}\right)^{\alpha} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)\, t^n. \tag{1.9}$$

Some of the explicit formulas for $p_n(x)$ are

$$\begin{aligned} p_n(x) &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^{n-k}(x+1)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{k}\left(\frac{1-x}{2}\right)^k \\ &= 2^n\sum_{k=0}^{n}\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}, \quad (\text{see [2, 34]}). \end{aligned}$$

Legendre polynomials occur in the solution of Laplacian equation of the static potential $\nabla^2\phi(x) = 0$, in a charge-free region of space, using the method of separation of variables, where the boundary condition has axial symmetry (no dependence on an azimuthal angle).

Where $\hat{z}$ is the axis of symmetry and θ is the angle between the position of the observer and the $\hat{z}$ axis (the zenith angle), the solution for the potential will be

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] p_l(\cos\theta).$$

Here A_l and B_l are to be determined according to the boundary condition of each problem (see [21]).

Recently, several authors have studied some interesting extensions and modifications of Legendre polynomials along with related combinatorial, probabilistic, physics, and physical applications (see [1–33]).

Kim in [25, 26], and Kim-Kim in [23] considered some non-linear differential equations arising from special numbers and polynomials and derived some new and interesting combinatorial identities.

In this paper, we consider some differential equations arising from the generating function of Legendre polynomials and give some new and explicit identities on the Legendre polynomials which are derived from the solutions of our differential equations.

2. Differential equations arising from Legendre polynomials

Let

$$F = F(t,x) = \frac{1}{\sqrt{1-2tx+t^2}}. \tag{2.1}$$

Then

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x) = \left(-\frac{1}{2}\right)\left(1-2tx+t^2\right)^{-\frac{3}{2}}(-2x+2t) \\ &= (x-t)\left(1-2tx+t^2\right)^{-\frac{3}{2}} \\ &= (x-t)F^3. \end{aligned} \tag{2.2}$$

From (2.2), we have

$$F^3 = \frac{1}{x-t}F^{(1)}. \tag{2.3}$$

For $N \in \mathbb{N}$, let

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x), \quad F^N = \underbrace{F \times \cdots \times F}_{N-\text{times}}. \tag{2.4}$$

From (2.3), we can derive the following equations:

$$3F^2F^{(1)} = \frac{(-1)^2}{(x-t)^2}F^{(1)} + \frac{1}{x-t}F^{(2)}. \tag{2.5}$$

By (2.2) and (2.5), we get

$$3F^5 = \frac{1}{(x-t)^3}F^{(1)} + \frac{1}{(x-t)^2}F^{(2)}. \tag{2.6}$$

From (2.6), we have

$$\begin{aligned} 3\cdot 5F^4F^{(1)} &= \frac{3}{(x-t)^4}F^{(1)} + \frac{1}{(x-t)^3}F^{(2)} + \frac{2}{(x-t)^3}F^{(2)} + \frac{1}{(x-t)^2}F^{(3)} \\ &= \frac{3}{(x-t)^4}F^{(1)} + \frac{3}{(x-t)^3}F^{(2)} + \frac{1}{(x-t)^2}F^{(3)}. \end{aligned} \tag{2.7}$$

From (2.2) and (2.7), we get

$$3\cdot 5F^7 = \frac{3}{(x-t)^5}F^{(1)} + \frac{3}{(x-t)^4}F^{(2)} + \frac{1}{(x-t)^3}F^{(3)}. \tag{2.8}$$

From (2.8), we note that

$$3\cdot 5\cdot 7F^{6}F^{(1)}=\frac{5\cdot 3}{(x-t)^{6}}F^{(1)}+\frac{3}{(x-t)^{5}}F^{(2)}+\frac{4\cdot 3}{(x-t)^{5}}F^{(2)}+\frac{3}{(x-t)^{4}}F^{(3)}+\frac{3}{(x-t)^{4}}F^{(3)}+\frac{1}{(x-t)^{3}}F^{(4)}. \tag{2.9}$$

Thus, by (2.2) and (2.9), we get

$$3\cdot 5\cdot 7F^{9}=\frac{5\cdot 3}{(x-t)^{7}}F^{(1)}+\frac{5\cdot 3}{(x-t)^{6}}F^{(3)}+\frac{3\cdot 2}{(x-t)^{5}}F^{(3)}+\frac{1}{(x-t)^{4}}F^{(4)}. \tag{2.10}$$

Continuing this process, we set

$$(2N-1)!!F^{2N+1}=\sum_{i=1}^{N}a_i(N)\frac{F^{(i)}}{(x-t)^{2N-i}}. \tag{2.11}$$

Thus, by (2.11), we get

$$\begin{aligned}&(2N-1)!!\,(2N+1)\,F^{2N}F^{(1)}\\&=\sum_{i=1}^{N}a_i(N)\frac{d}{dt}\left(\frac{F^{(i)}}{(x-t)^{2N-i}}\right)\\&=\sum_{i=1}^{N}a_i(N)\left\{\frac{2N-i}{(x-t)^{2N-i+1}}F^{(i)}+\frac{1}{(x-t)^{2N-i}}F^{(i+1)}\right\}\\&=\sum_{i=1}^{N}a_i(N)\frac{2N-i}{(x-t)^{2N-i+1}}F^{(i)}+\sum_{i=1}^{N}a_i(N)\frac{F^{(i+1)}}{(x-t)^{2N-i}}.\end{aligned} \tag{2.12}$$

From (2.2) and (2.12), we have

$$\begin{aligned}&(2N+1)!!F^{2N+3}\\&=\sum_{i=1}^{N}a_i(N)\frac{2N-i}{(x-t)^{2(N+1)-i}}F^{(i)}+\sum_{i=1}^{N}a_i(N)\frac{F^{(i+1)}}{(x-t)^{2N-i+1}}\\&=\sum_{i=1}^{N}a_i(N)\frac{2N-i}{(x-t)^{2(N+1)-i}}F^{(i)}+\sum_{i=1}^{N}a_i(N)\frac{F^{(i+1)}}{(x-t)^{2N-i+1}}\\&=(2N-1)\,a_1(N)\frac{F^{(1)}}{(x-t)^{2N+1}}+a_N(N)\frac{F^{(N+1)}}{(x-t)^{N+1}}\\&\quad+\sum_{i=2}^{N}\{(2N-i)\,a_i(N)+a_{i-1}(N)\}\frac{F^{(i)}}{(x-t)^{2(N+1)-i}}.\end{aligned} \tag{2.13}$$

By replacing N by $N+1$ in (2.11), we get

$$(2N+1)!!F^{2N+3}=\sum_{i=1}^{N+1}a_i(N+1)\frac{F^{(i)}}{(x-t)^{2(N+1)-i}}. \tag{2.14}$$

By comparing the coefficients on the both sides of (2.13) and (2.14), we have

$$a_1(N+1)=(2N-1)\,a_1(N), \tag{2.15}$$

$$a_{N+1}(N+1)=a_N(N), \tag{2.16}$$

and

(2.17) $$a_i(N+1) = (2N-i)\,a_i(N) + a_{i-1}(N), \quad (2 \le i \le N).$$

From (2.3) and (2.11), we note that

(2.18) $$\frac{1}{x-t}F^{(1)} = F^3 = a_1(1)\,\frac{1}{x-t}F^{(1)}.$$

Thus, by comparing the coefficients on both sides of (2.18), we get $a_1(1) = 1$.

From (2.15) and (2.16), we can derive the following equations:

(2.19) $$\begin{aligned} a_1(N+1) &= (2N-1)\,a_1(N) \\ &= (2N-1)(2N-3)\,a_1(N-1) \\ &\;\;\vdots \\ &= (2N-1)(2N-3)\cdots 3\cdot 1 a_1(1) \\ &= (2N-1)!!, \end{aligned}$$

and

(2.20) $$a_{N+1}(N+1) = a_N(N) = a_{N-1}(N-1) = \cdots = a_1(1) = 1.$$

By (2.17), for $2 \le i \le N$, we get

(2.21)
$$\begin{aligned} &a_i(N+1) \\ &= (2N-i)\,a_i(N) + a_{i-1}(N) \\ &= (2N-i)\{(2(N-1)-i)\,a_i(N-1) + a_{i-1}(N-1)\} + a_{i-1}(N) \\ &= (2N-i)(2N-2-i)\,a_i(N-1) + (2N-i)\,a_{i-1}(N-1) + a_{i-1}(N) \\ &= (2N-i)(2N-2-i)(2N-4-i)\,a_i(N-2) + (2N-i)(2N-2-i)\,a_{i-1}(N-2) \\ &\quad + (2N-i)\,a_{i-1}(N-1) + a_{i-1}(N) \\ &\vdots \\ &= \left(\prod_{l=0}^{N-i}(2N-2l-i)\right)a_i(i) + \sum_{l=0}^{N-i}\langle 2N-i\rangle_l\, a_{i-1}(N-l) \\ &= \prod_{l=0}^{N-i}(2N-2l-i) + \sum_{l=0}^{N-i}\langle 2N-i\rangle_l\, a_{i-1}(N-l), \end{aligned}$$

where $\langle 2N+\alpha\rangle_k = (2N+\alpha)(2(N-1)+\alpha)\cdots(2(N-k+1)+\alpha)$, and $\langle 2N+\alpha\rangle_0 = 1$.

From (2.21), we have

(2.22) $$\begin{aligned} &a_{i-1}(N-l_1) \\ &= \prod_{l_2=0}^{N-l_1-i}(2N-2l_1-2l_2-i-1) \\ &\quad + \sum_{l_2=0}^{N-l_1-i}\langle 2N-2l_1-i-1\rangle_{l_2}\, a_{i-2}(N-l_1-l_2-1). \end{aligned}$$

By (2.21) and (2.22), we get

$$\begin{aligned} (2.23)\quad & a_i(N+1) \\ &= \prod_{l_1=0}^{N-i}(2N-2l_1-i) + \sum_{l_1=0}^{N-i}\langle 2N-i\rangle_{l_1} a_{i-1}(N-l_1) \\ &= \prod_{l_1=0}^{N-i}(2N-2l_1-i) + \sum_{l_1=0}^{N-i}\left(\prod_{l_2=0}^{N-l_1-i}(2N-2l_1-2l_2-i-1)\right)\langle 2N-i\rangle_{l_1} \\ &\quad + \sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\langle 2N-i\rangle_{l_1}\langle 2N-2l_1-i-1\rangle_{l_2} a_{i-2}(N-l_1-l_2-1). \end{aligned}$$

Now, we observe that

$$\begin{aligned} (2.24)\quad & a_{i-2}(N-l_1-l_2-1) \\ &= a_{i-2}(N-l_1-l_2-2+1) \\ &= \prod_{l_3=0}^{N-l_1-l_2-i}(2N-2l_1-2l_2-2l_3-i+2) \\ &\quad + \sum_{l_3=0}^{N-l_1-l_2-i}\langle 2N-2l_1-2l_2-i-2\rangle_{l_3} a_{i-3}(N-l_1-l_2-l_3-2). \end{aligned}$$

From (2.23) and (2.24), and continuing this process, we obtain

$$\begin{aligned} & a_i(N+1) \\ &= \prod_{l_1=0}^{N-i}(2N-2l_1-i) \\ &\quad + \sum_{l_1=0}^{N-i}\langle 2N-i\rangle_{l_1}\left(\prod_{l_2=0}^{N-l_1-i}(2N-2l_1-2l_2-i-1)\right) \\ &\quad + \sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\langle 2N-i\rangle_{l_1}\langle 2N-2l_1-i-1\rangle_{l_2} \\ &\quad \times\left(\prod_{l_3=0}^{N-l_1-l_2-i}(2N-2l_1-2l_2-2l_3-i-2)\right)+\cdots \\ &\quad + \sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\cdots\sum_{l_{i-2}=0}^{N-l_1-\cdots-l_{i-3}-i}\langle 2N-i\rangle_{l_1} \\ &\quad \times\langle 2N-2l_1-i-1\rangle_{l_2}\cdots\langle 2N-2l_1-\cdots-2l_{i-3}-2i+3\rangle_{l_{i-2}} \\ &\quad \times\left(\prod_{l_{i-1}=0}^{N-l_1-\cdots-l_{i-2}-i}(2N-2l_1-2l_2-\cdots-2l_{i-1}-2i+2)\right) \\ &\quad + \sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\cdots\sum_{l_{i-1}=0}^{N-l_1-l_2-\cdots-l_{i-2}-i}\langle 2N-i\rangle_{l_1} \end{aligned}$$

$$\times \langle 2N-2l_1-i-1\rangle_{l_2}\cdots\langle 2N-2l_1-2l_2-\cdots-2l_{i-2}-2i+2\rangle_{l_{i-1}}$$
$$\times a_1\left(N-l_1-l_2-\cdots-l_{i-1}-i+2\right)$$
$$=\prod_{l_1=0}^{N-i}\left(2N-2l_1-i\right)$$
$$+\sum_{l_1=0}^{N-i}\langle 2N-i\rangle_{l_1}\left(\prod_{l_2=0}^{N-l_1-i}\left(2N-2l_1-2l_2-i-1\right)\right)$$
$$+\sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\langle 2N-i\rangle_{l_1}\langle 2N-2l_1-i-1\rangle_{l_2}$$
$$\times\left(\prod_{l_3=0}^{N-l_1-l_2-i}\left(2N-2l_1-2l_2-2l_3-i-2\right)\right)+\cdots$$
$$+\sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\cdots\sum_{l_{i-2}=0}^{N-l_1-\cdots-l_{i-3}-i}\langle 2N-i\rangle_{l_1}$$
$$\times\langle 2N-2l_1-i-1\rangle_{l_2}\cdots\langle 2N-2l_1-\cdots-2l_{i-3}-2i+3\rangle_{l_{i-2}}$$
$$\times\left(\prod_{l_{i-1}=0}^{N-l_1-\cdots-l_{i-2}-i}\left(2N-2l_1-2l_2-\cdots-2l_{i-1}-2i+2\right)\right)$$
$$+\sum_{l_1=0}^{N-i}\sum_{l_2=0}^{N-l_1-i}\cdots\sum_{l_{i-1}=0}^{N-l_1-l_2-\cdots-l_{i-2}-i}\langle 2N-i\rangle_{l_1}$$
$$\times\langle 2N-2l_1-i-1\rangle_{l_2}\cdots\langle 2N-2l_1-2l_2-\cdots-2l_{i-2}-2i+2\rangle_{l_{i-1}}$$
$$\times\left(2\left(N-l_1-l_2-\cdots-l_{i-1}-i\right)+1\right)!!.$$

Therefore, we obtain the following theorem.

Theorem 1. *The following non-linear differential equations*

$$(2N-1)!!F^{2N+1}=\sum_{i=1}^{N}a_i\left(N\right)\frac{F^{(i)}}{\left(x-t\right)^{2N-i}},\quad\left(N=1,2,\dots\right),$$

has a solution

$$F=F\left(t,x\right)=\frac{1}{\sqrt{1-2tx+t^2}},$$

where $a_1\left(N\right)=(2N-3)!!$, $a_N\left(N\right)=1$, *and, for* $2\le i\le N-1$,

$$a_i\left(N\right)$$
$$=\prod_{l_1=0}^{N-1-i}\left(2N-2l_1-i-2\right)$$
$$+\sum_{l_1=0}^{N-i-1}\langle 2N-2-i\rangle_{l_1}\left(\prod_{l_2=0}^{N-l_1-i-1}\left(2N-2l_1-2l_2-i-3\right)\right)$$
$$+\sum_{l_1=0}^{N-i-1}\sum_{l_2=0}^{N-l_1-i-1}\langle 2N-i-2\rangle_{l_1}\langle 2N-2l_1-i-3\rangle_{l_2}$$

$$\times \left(\prod_{l_3=0}^{N-l_1-l_2-i-1} (2N - 2l_1 - 2l_2 - 2l_3 - i - 4) \right) + \cdots$$
$$+ \sum_{l_1=0}^{N-i-1} \sum_{l_2=0}^{N-l_1-i-1} \cdots \sum_{l_{i-2}=0}^{N-l_1-\cdots-l_{i-3}-i-1} \langle 2N - i - 2 \rangle_{l_1}$$
$$\times \langle 2N - 2l_1 - i - 3 \rangle_{l_2} \cdots \langle 2N - 2l_1 - \cdots - 2l_{i-3} - 2i + 1 \rangle_{l_{i-2}}$$
$$\times \left(\prod_{l_{i-1}=0}^{N-l_1-\cdots-l_{i-2}-i-1} (2N - 2l_1 - 2l_2 - \cdots - 2l_{i-1} - 2i) \right)$$
$$+ \sum_{l_1=0}^{N-i-1} \sum_{l_2=0}^{N-l_1-i-1} \cdots \sum_{l_{i-1}=0}^{N-l_1-l_2-\cdots-l_{i-2}-i-1} \langle 2N - i - 2 \rangle_{l_1}$$
$$\times \langle 2N - 2l_1 - i - 3 \rangle_{l_2} \cdots \langle 2N - 2l_1 - 2l_2 - \cdots - 2l_{i-2} - 2i \rangle_{l_{i-1}}$$
$$\times (2 (N - l_1 - l_2 - \cdots - l_{i-1} - i) - 1)!!.$$

Recall that the generating function of Legendre polynomials is given by

$$F = F(t, x) = \frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} p_n(x) t^n. \tag{2.25}$$

Thus, by (2.25), we get

$$F^{(i)} = \left(\frac{d}{dt} \right)^i F(t, x) = \sum_{n=i}^{\infty} p_n(x) (n)_i t^{n-i} = \sum_{n=0}^{\infty} p_{n+i}(x) (n+i)_i t^n, \tag{2.26}$$

where $(x)_n = x(x-1) \cdots (x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

From (1.9), we note that

$$F^{2N+1} = \sum_{n=0}^{\infty} p_n^{(2N+1)}(x) t^n. \tag{2.27}$$

By Theorem 1, we get

(2.28)

$$F^{2N+1}$$
$$= \frac{1}{(2N-1)!!} \sum_{i=1}^{N} a_i(N) \frac{F^{(i)}}{(x-t)^{2N-i}}$$
$$= \frac{1}{(2N-1)!!} \sum_{i=1}^{N} a_i(N) \left(\sum_{m=0}^{\infty} \binom{2N+m-i-1}{m} x^{-(2N+m-i)} t^m \right)$$
$$\times \left(\sum_{l=0}^{\infty} p_{l+i}(x) (l+i)_i t^l \right)$$
$$= \frac{1}{(2N-1)!!} \sum_{i=1}^{N} a_i(N)$$

$$\times \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{2N+m-i-1}{m} x^{-(2N+m-i)} p_{n-m+i}(x) (n-m+i)_i \right\} t^n.$$

Therefore, by (2.27) and (2.28), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$p_n^{(2N+1)}(x) = \frac{1}{(2N-1)!!} \sum_{i=1}^{N} \sum_{m=0}^{n} a_i(N) \binom{2N+m-i-1}{m} \times x^{-(2N+m-i)} p_{n-m+i}(x) (n-m+i)_i,$$

where $a_i(N)$ $(1 \leq i \leq N)$ are as in Theorem 1.

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